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Measure-valued branching processes with immigration

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Starting from the cumulant semigroup of a measure-valued branching process, we construct the transition probabilities of some Markov process $Y^{(\beta)} = (Y_t^{(\beta)}, t \in \mathbb{R})$, which we call a measure-valued branching process with discrete immigration of unit β. The immigration of $Y^{(\beta)}$ is governed by a Poisson random measure ρ on the time-distribution space and a probability generating function h, both depending on β. It is shown that, under suitable hypotheses, $Y^{(\beta)}$ approximates to a Markov process $Y = (Y_t, t \in \mathbb{R})$ as $\beta \to 0^+$. The latter is the one we call a measure-valued branching process with immigration. The convergence of branching particle systems with immigration is also studied.

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measure-valued branching process * immigration * particle system * superprocess * weak convergence

1. Introduction

Let M be the totality of finite measures on a measurable space $(E, \mathscr{E})$. Suppose that $X = (X_t, t \in \mathbb{R})$ is a Markov process in M with transition function $P(r, \mu; t, \mathrm{d}\nu)$. X is called a measure-valued branching process (MB-process) if

$$P(r, \mu_1 + \mu_2; t, \cdot) = P(r, \mu_1; t, \cdot) * P(r, \mu_2; t, \cdot), \quad \mu_1, \mu_2 \in M, \ r \leq t, \tag{1.1}$$

where '*' denotes the convolution operation (cf. Dawson, 1977, Dawson and Ivanoff, 1978, Watanabe, 1968, etc). When E is reduced to one point, X takes values in $\mathbb{R}^+ := [0, \infty)$ and is called a continuous state branching process (CB-process).

Continuous state branching processes with immigration (CBI-processes) were first introduced by Kawazu and Watanabe (1971). Several authors have also studied measure-valued branching processes with immigration (MBI-processes); see Dynkin (1990, 1991), Konno and Shiga (1988), etc.

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In the present paper, we study a general class of MBI-processes that covers the models of the previous authors and can be regarded as the measure-valued counterpart of the one of CBI-processes proposed by Kawazu and Watanabe. Section 2 contains some preliminaries. The general definition for an MBI-process is given in Section 3, followed by the model of a measure-valued branching process with discrete immigration (MBDI-process). The heuristic meanings of the latter are clear. It is shown that the MBI-process is in fact an approximation for the MBDI-process with high rate and small unit of immigration. In Section 4, we study the convergence of branching particle systems with immigration to MBI-processes. A branching system of particles with immigration is not an MBDI-process in the terminology of this paper. The concluding Section 5 contains a brief discussion of MBI-processes with σ-finite values whose study can be reduced to that of the class with finite values studied in Sections 3 and 4.

2. Preliminaries

2.1. We first introduce some notation. If F is a topological space, then $\mathscr{B}(F)$ denotes the σ-algebra of F generated by all open sets, and

$B(F)=\{$bounded $\mathscr{B}(F)$-measurable functions on $F\}$,

$C(F)=\{f: f\in B(F)$ is continuous$\}$,

$B(F)_a=\{f: f\in B(F)$ and $\|f\|\leq a\}$ for $a\geq 0$.

Here '$\|\cdot\|$' denotes the supremum norm. In the case F is locally compact,

$C_0(F)=\{f: f\in C(F)$ vanishes at infinity$\}$.

The subsets of nonnegative members of the function spaces are denoted by the superscript '+', and those of strictly positive members by '++'; e.g., $B(F)^+$, $C(F)^{++}$. If F is a metric space, then $D(\mathbb{R}^+, F)$ stands for the space of cadlag functions from $\mathbb{R}^+$ to F equipped with the Skorohod topology. Finally, δ_x denotes the unit mass concentrated at x, and for a function f and a measure μ, $\langle\mu,f\rangle=\int f\,\mathrm{d}\mu$.

2.2. Suppose that E is a topological Lusin space, i.e., a homeomorph of a Borel subset of some compact metric space. Let

$M=\{$finite measures on $(E,\mathscr{B}(E))\}$,

$M_0=\{\pi\colon \pi\in M$ and $\pi(E)=1\}$,

$M_1=\{\sigma\colon \sigma\in M$ is integer-valued$\}$,

$M_k=\{k^{-1}\sigma\colon \sigma\in M_1\}$ for $k=2,3,\ldots$.

We topologize M, and hence M_k, $k=0,1,2,\ldots$, with the weak convergence topology. It is well known that M is locally compact and separable when E is a compact metric space.

The Laplace functional of a probability measure P on M is defined as

$$L_P(f)=\int_M \mathrm{e}^{-\langle\mu,f\rangle}P(\mathrm{d}\mu),\quad f\in B(E)^+. \tag{2.1}$$

P is said to be infinitely divisible if for each integer $m > 0$, there is a probability measure P_m on M such that $L_P(f) = [L_{P_m}(f)]^m$.

We say a functional w on $B(E)^+$ belongs to the *class* $\mathscr{W}$ if it has the representation

$$w(f) = \iint_{\mathbb{R}^+ \times M_0} (1 - e^{-u\langle \pi, f\rangle}) \frac{1+u}{u} G(du, d\pi), \quad f \in B(E)^+, \tag{2.2}$$

where G is a finite measure on $\mathbb{R}^+ \times M_0$ and the value of the integrand at $u = 0$ is defined as $\langle \pi, f\rangle$. The following result coincides with Theorem 1.2 of Watanabe (1968) since $(E, \mathscr{B}(E))$ is isomorphic to a compact metric space with the Borel σ-algebra.

Proposition 2.1. *A probability measure P on M is infinitely divisible if and only if* $-\log L_P(\cdot) \in \mathscr{W}$. $\square$

A family of operators $W_t^r : f \mapsto w_t^r(\cdot, f)$ $(r \leq t \in \mathbb{R})$ on $B(E)^+$ is called a *cumulant semigroup* provided

(2.A) for every fixed $r \leq t$ and x, $w_t^r(x, \cdot)$ belongs to $\mathscr{W}$;

(2.B) for all $r \leq s \leq t$, $W_s^r W_t^s = W_t^r$ and $W_r^r f \equiv f$.

We say the cumulant semigroup is *homogeneous* if $W_t^r = W_{t-r}$ only depends on the difference $t - r \geq 0$. A homogeneous cumulant semigroup $W_t, t \geq 0$, is called a *Ψ-semigroup* provided E is a compact metric space and W_t preserves $C(E)^{++}$ for all $t \geq 0$ (cf. Watanabe, 1968).

2.3.

Definition 2.2. Suppose that $X = (X_t, P_{r,\mu})$ is an MB-process in the space M. Let

$$w_t^r(x) \equiv w_t^r(x, f) = -\log P_{r,\delta_x} \exp\langle X_t, -f\rangle. \tag{2.3}$$

We say X is *regular* if for every $f \in B(E)^+$ and $r \leq t$, the function $w_t^r(\cdot)$ belongs to $B(E)^+$ and

$$P_{r,\mu} \exp\langle X_t, -f\rangle = \exp\langle \mu, -w_t^r\rangle, \quad \mu \in M. \tag{2.4}$$

Here $P_{r,\mu}$ denotes the conditional expectation given $X_r = \mu$.

An easy application of Proposition 2.1 gives the following:

Proposition 2.3. *Formula* (2.4) *defines the transition probabilities of a regular MB-process $X = (X_t, P_{r,\mu})$ if and only if $W_t^r : f \mapsto w_t^r$ is a cumulant semigroup.* $\square$

If $W_t : f \mapsto w_t$ is a homogeneous cumulant semigroup, then

$$P_\mu \exp\langle X_t, -f\rangle = \exp\langle \mu, -w_t\rangle \tag{2.5}$$

determines the transition probabilities of a homogeneous MB-process $X=(X_t, P_\mu)$. In the case E is a compact metric space, Watanabe (1968) showed that a homogeneous MB-process is a Feller process if and only if it is regular and the corresponding cumulant semigroup is a Ψ-semigroup.

2.4. A special form of the MB-process is the 'superprocess' that arises as the high density limit of a branching particle system. Suppose that

(2.C) $\xi=(\Omega, \mathscr{F}, \mathscr{F}_t^r, \xi_t(\omega), \Pi_{r,x})$ is a Markov process in the space E with right continuous sample paths and Borel measurable transition probabilities, i.e., for every $f\in B(E)$ and $t\in\mathbb{R}$ the function $1_{\{r\leq t\}}\Pi_{r,x}f(\xi_t)$ is measurable in (r,x);

(2.D) $K=K(\omega,t)$ is a continuous additive functional of ξ such that $\sup_\omega|K(\omega,t)|<\infty$ for every $t\in\mathbb{R}$;

(2.E) $\phi=\phi^s(x,\lambda)$ is a $\mathscr{B}(\mathbb{R}\times E\times\mathbb{R}^+)$-measurable function given by

$$\phi^s(x,\lambda)=b^s(x)\lambda+c^s(x)\lambda^2+\int_0^\infty(\mathrm{e}^{-\lambda u}-1+\lambda u)m^s(x,\mathrm{d}u),$$

where $c^s(x)$ is nonnegative, $m^s(x,\cdot)$ is carried by $(0,\infty)$, and the function

$$|b^s(x)|+c^s(x)+\int_0^\infty u\wedge u^2 m^s(x,\mathrm{d}u)$$

of (s,x) is bounded on $\mathbb{R}\times E$.

A regular MB-process $X=(X_t, P_{r,\mu})$ is called a (ξ, K, ϕ)*-superprocess* if it has the cumulant semigroup $f\mapsto w_t^r$ determined by the evolution equation

$$w_t^r(x)+\Pi_{r,x}\int_r^t\phi^s(\xi_s, w_t^s(\xi_s))K(\mathrm{d}s)=\Pi_{r,x}f(\xi_t),\quad r\leq t. \tag{2.6}$$

The existence and the uniqueness of the solution to the above equation have been proved by Dynkin (1990, 1991). Note that the hypothesis $\int_0^\infty u\wedge u^2m(\mathrm{d}s)<\infty$ makes things work only for the MB-processes with finite first moments. (Dynkin also assumed $b^s(x)\geq 0$ for (2.E), but this restriction is not essential; see Section 4 of this paper.)

3. MBI-processes

3.1.

Definition 3.1. Let E be a topological Lusin space. Suppose that

(3.A) $W_t^r: f\mapsto w_t^r$ $(r\leq t\in\mathbb{R})$ is a cumulant semigroup such that for every $f\in B(E)^+$ and $u\leq t\in\mathbb{R}$, the function $w_t^r(x)$ of (r,x) restricted to $[u,t]\times E$ belongs to $B([u,t]\times E)^+$;

(3.B) H is a measure on $\mathbb{R}\times M_0$ such that $H([u,t]\times M_0)<\infty$ for every $u\leq t\in\mathbb{R}$;

(3.C) $\psi^s(\pi,\lambda)$ is a $\mathscr{B}(\mathbb{R}\times M_0\times\mathbb{R}^+)$-measurable function given by

$$\psi^s(\pi,\lambda)=d^s(\pi)\lambda+\int_0^\infty(1-\mathrm{e}^{-\lambda u})n^s(\pi,\mathrm{d}u),\quad s\in\mathbb{R},\ \pi\in M_0,\ \lambda\in\mathbb{R}^+,$$

where $d^s(\pi)$ is nonnegative, $n^s(\pi, \cdot)$ is carried by $(0, \infty)$, and

$$\sup_{s,\pi}\left[d^s(\pi)+\int_0^\infty 1\wedge u\, n^s(\pi, \mathrm{d}u)\right]<\infty.$$

A Markov process $Y=(Y_t, Q_{r,\mu})$ in the space M is called an *MBI-process with parameters* (W, H, ψ) if

$$Q_{r,\mu}\exp\langle Y_t, -f\rangle=\exp\left\{-\langle\mu, w_t^r\rangle-\iint_{(r,t]\times M_0}\psi^s(\pi, \langle\pi, w_t^s\rangle)H(\mathrm{d}s, \mathrm{d}\pi)\right\} \tag{3.1}$$

for $f\in B(E)^+$, $\mu\in M$ and $r\leqslant t$.

Remark 3.2. (i) That the right-hand side of (3.1) is indeed a Laplace transform follows once we observe that the functional is positive definite on semigroup $B(E)^+$. (See Berg et al., 1984, and Fitzsimmons, 1988, for details on positive definite functionals.) This fact also follows from the proof of Theorem 3.5 in Section 3.3.

(ii) We call the MBI-process defined by (3.1) a *(ξ, K, ϕ, H, ψ)-superprocess* if the corresponding cumulant semigroup $f\mapsto w_t^r$ is determined by equation (2.6). Dynkin (1990, 1991) has studied the (ξ, K, ϕ, H, ψ)-superprocess in the case where H is carried by $\mathbb{R}\times\{\delta_x: x\in E\}$ and $\psi^s(\pi, \lambda)\equiv\lambda$.

A time homogeneous MBI-process $Y=(Y_t, Q_\mu)$ is determined by three parameters (W, η, ψ):

$$Q_\mu\exp\langle Y_t, -f\rangle=\exp\left\{-\langle\mu, w_t\rangle-\int_0^t \mathrm{d}s\int_{M_0}\psi(\pi, \langle\pi, w_s\rangle)\eta(\mathrm{d}\pi)\right\}, \tag{3.2}$$

where $W_t: f\mapsto w_t$ is a homogeneous cumulant semigroup, η is a finite measure on M_0, and $\psi=\psi(\pi, \lambda)$, given by (3.C), does not depend on s. Note that if W_t is a strongly continuous Ψ-semigroup on $C(E)^{++}$, then the process Y has a strongly continuous Feller semigroup on $C_0(M)$, so it has a version in $D(\mathbb{R}^+, M)$ (see, for example, Ethier and Kurtz, 1986).

Example 3.3. When E is reduced to one point, the MBI-process takes values in $\mathbb{R}^+$ and is called a CBI-process. In this case (3.2) becomes

$$Q_\mu\,\mathrm{e}^{-zY_t}=\exp\left\{-\mu w_t-\int_0^t\psi(w_s)\,\mathrm{d}s\right\},\quad z\geqslant 0,\ \mu\geqslant 0,\ t\geqslant 0. \tag{3.3}$$

Kawazu and Watanabe (1971) showed that if the process Y is stochastically continuous for every Q_μ, then w_t satisfies

$$\frac{\mathrm{d}w_t}{\mathrm{d}t}=-\phi(w_t),\quad w_0=z, \tag{3.4}$$

for a function ϕ with the representation

$$\phi(\lambda)=b\lambda+c\lambda^2+\int_0^\infty\left(e^{-\lambda u}-1+\frac{\lambda u}{1+u^2}\right)m(du), \tag{3.5}$$

where $c\geq 0$ and $\int_0^\infty 1\wedge u^2 m(du)<\infty$.

3.2. An MBDI-process $Y=(Y_t, t\in\mathbb{R})$ depends on four parameters (W, H, h, β), where W and H are given by (3.A) and (3.B), β is a positive number, and

(3.D) $h^s(\pi, z)=\sum_{i=0}^\infty q_i^s(\pi)z^i$, for every $(s,\pi)\in\mathbb{R}\times M_0$, is a probability generating function with all $q_i=q_i^s(\pi)$ measurable in (s,π).

Such a process is characterized by the following properties:

(i) the evolution of the branch $(X_t, t\geq r)$ of Y with $X_r=\mu$ a.s. is determined by the Laplace functional (2.4);

(ii) the entry times and entry distributions of the immigrants are governed by a Poisson random measure ρ on the product space $\mathbb{R}\times M_0$ with intensity $H(ds, d\pi)$;

(iii) the generating function $h^s(\pi,\cdot)$ describes the number of drops, each of those having mass β, entering E at time s with distribution $\pi(dx)$.

We refer to β as the immigration unit. Suppose that different drops of the immigrants land in E independently of each other and that the immigration is independent of the inner population. Then the MBDI-process is a Markov process in space M. Let $Q_{r,\mu}$ denote the conditional law of $(Y_t, t\geq r)$ given $Y_r=\mu$, and let D denote the distribution of the random measure ρ on space

$$\left\{\zeta\equiv\sum_{\alpha=1}^{\zeta(\mathbb{R}\times M_0)}\delta_{(s_\alpha,\pi_\alpha)}:(s_\alpha,\pi_\alpha)\in\mathbb{R}\times M_0\right\}.$$

Properties (i)-(iii) lead through a calculation to the Laplace functional:

$$\begin{aligned}
&Q_{r,\mu}\exp\langle Y_t,-f\rangle\\
&=\exp\langle\mu,-w_t^r\rangle\int D(d\zeta)\prod_{r<s_\alpha\leq t}\sum_{i=0}^\infty q_i^{s_\alpha}(\pi_\alpha)\langle\pi_\alpha,\exp\{-\beta w_t^{s_\alpha}\}\rangle^i\\
&=\exp\langle\mu,-w_t^r\rangle\int D(d\zeta)\exp\iint_{(r,t]\times M_0}\log h^s(\pi,\langle\pi,e^{-\beta w_t^s}\rangle)\zeta(ds,d\pi)\\
&=\exp\left\{-\langle\mu,w_t^r\rangle-\iint_{(r,t]\times M_0}[1-h^s(\pi,\langle\pi,e^{-\beta w_t^s}\rangle)]H(ds,d\pi)\right\}.
\end{aligned}\tag{3.6}$$

3.3. Consider a sequence of MBDI-processes $Y^{(k)}=(Y_t^{(k)}, Q_{r,\mu}^{(k)})$ with parameters $(W, \alpha_k H, h_k, k^{-1})$, where $\alpha_k\geq 0$, $k=1,2,\ldots$. By (3.6) we have

$$\begin{aligned}
&Q_{r,\mu}^{(k)}\exp\langle Y_t^{(k)},-f\rangle\\
&=\exp\left\{-\langle\mu,w_t^r\rangle-\iint_{(r,t]\times M_0}\psi_k^s(\pi,\langle\pi,w_t^s(k)\rangle)H(ds,d\pi)\right\},
\end{aligned}\tag{3.7}$$

where

$$w_t^s(k,x)=k[1-\exp\{-k^{-1}w_t^s(x)\}], \tag{3.8}$$

and

$$\psi_k^s(\pi,\lambda)=\alpha_k[1-h_k^s(\pi,1-\lambda/k)],\quad 0\leqslant\lambda\leqslant k. \tag{3.9}$$

Since $w_t^s(k)\to w_t^s$ as $k\to\infty$, it is natural to assume the sequence ψ_k to converge if one hopes to obtain $Y_t=\lim_{k\to\infty}Y_t^{(k)}$ in some sense.

Lemma 3.4. (i) *Suppose that*

(3.E) $\psi_k^s(\pi,\lambda)\to\psi^s(\pi,\lambda)$ $(k\to\infty)$ *boundedly and uniformly on the set* $\mathbb{R}\times M_0\times[0,l]$ *of* (s,π,λ) *for each* $l\geqslant 0$.

Then $\psi^s(\pi,\lambda)$ *has the representation* (3.C).

(ii) *To each function* ψ *given by* (3.C) *there corresponds a sequence in form* (3.9) *such that*

$$\psi_k^s(\pi,\lambda)=\psi^s(\pi,\lambda),\quad s\in\mathbb{R},\ \pi\in M_0,\ 0\leqslant\lambda\leqslant k.$$

Proof. Assertion (i) was proved in Li (1991). To get (ii) one can set

$$\alpha_k=1+\sup_{s,\pi}\left[kd^s(\pi)+\int_0^\infty(1-\mathrm{e}^{-ku})n^s(\pi,\mathrm{d}u)\right]$$

and

$$h_k^s(\pi,z)=1+k\alpha_k^{-1}d^s(\pi)(z-1)+\alpha_k^{-1}\int_0^\infty(\mathrm{e}^{ku(z-1)}-1)n^s(\pi,\mathrm{d}u).\qquad\square$$

Condition (3.E) usually implies $\alpha_k\to\infty$. Thus the following theorem shows that the MBI-process is an approximation for the MBDI-process with high rate and small unit of immigration.

Theorem 3.5. (i) *Let* $Y^{(k)}$ *be as above, and let* Y *be the MBI-process defined by* (3.1). *If* (3.E) *holds, then for every* $\mu\in M$, $r\leqslant t_1<\cdots<t_n\in\mathbb{R}$ *and* $a\geqslant 0$,

$$Q_{r,\mu}^{(k)}\exp\sum_{i=1}^n\langle Y_{t_i}^{(k)},-f_i\rangle\to Q_{r,\mu}\exp\sum_{i=1}^n\langle Y_{t_i},-f_i\rangle\quad(k\to\infty) \tag{3.10}$$

uniformly in $f_1,\ldots,f_n\in B(E)_a^+$.

(ii) *For each MBI-process* Y *defined by* (3.1), *there is a sequence of MBDI-processes* $Y^{(k)}$ *such that* (3.10) *is satisfied.*

Proof. It suffices to show assertion (i) since (ii) follows immediately from (i) and Lemma 3.4. We do this by induction in n.

Fix $a \geq 0$ and $r \leq t \in \mathbb{R}$. By (3.A) and (3.8), $w_t^s(k, x, f) \to w_t^s(x, f)$ $(k \to \infty)$ boundedly and uniformly in $(s, x, f) \in [r, t] \times E \times B(E)_a^+$. Thus (3.E) yields

$$\iint_{(r,t] \times M_0} \psi_k^s(\pi, \langle \pi, w_t^s(k) \rangle) H(\mathrm{d}s, \mathrm{d}\pi)$$

$$\to \iint_{(r,t] \times M_0} \psi^s(\pi, \langle \pi, w_t^s \rangle) H(\mathrm{d}s, \mathrm{d}\pi) \quad (k \to \infty) \tag{3.11}$$

uniformly in $f \in B(E)_a^+$. To see that the right-hand side of (3.1) is indeed the Laplace functional of a probability measure we appeal to the following:

Lemma 3.6 (Kallenberg, 1983; Dynkin, 1991). *Suppose that P_k, $k = 1, 2, \ldots$, are probability measures on M. If $L_{P_k}(f) \to L(f)$ $(k \to \infty)$ uniformly in $f \in B(E)_a^+$ for every $a \geq 0$, then L is the Laplace functional of a probability measure on M.* □

Then it follows immediately that (3.1) really defines the transition probabilities of a Markov process Y in space M and that (3.10) holds for $n = 1$.

Now assuming (3.10) is true for $n = m$, we show the fact for $n = m + 1$. Let $r \leq t_1 < \cdots < t_{m+1} \in \mathbb{R}$ and $f_1, \ldots, f_{m+1} \in B(E)^+$. Then

$$Q_{r,\mu}^{(k)} \exp \sum_{i=1}^{m+1} \langle Y_{t_i}^{(k)}, -f_i \rangle$$

$$= Q_{r,\mu}^{(k)} Q_{r,\mu}^{(k)} \left\{ \prod_{i=1}^{m+1} \exp\langle Y_{t_i}^{(k)}, -f_i \rangle \;\middle|\; Y_t^{(k)}, t \leq t_m \right\}$$

$$= Q_{r,\mu}^{(k)} \prod_{i=1}^{m} \exp\langle Y_{t_i}^{(k)}, -f_i \rangle Q_{r,\mu}^{(k)} \{ \exp\langle Y_{t_{m+1}}^{(k)}, -f_{m+1} \rangle \mid Y_{t_m}^{(k)} \}$$

$$= Q_{r,\mu}^{(k)} \prod_{i=1}^{m} \exp\langle Y_{t_i}^{(k)}, -f_i \rangle \cdot \exp\langle Y_{t_m}^{(k)}, -w_{t_{m+1}}^{t_m}(f_{m+1}) \rangle$$

$$\cdot \exp\left\{ - \iint_{(t_m, t_{m+1}] \times M_0} \psi_k^s(\pi, \langle \pi, w_{t_{m+1}}^s(k, f_{m+1}) \rangle) H(\mathrm{d}s, \mathrm{d}\pi) \right\}$$

$$= Q_{r,\mu}^{(k)} \prod_{i=1}^{m-1} \exp\langle Y_{t_i}^{(k)}, -f_i \rangle \cdot \exp\langle Y_{t_m}^{(k)}, -f_m - w_{t_{m+1}}^{t_m}(f_{m+1}) \rangle$$

$$\cdot \exp\left\{ - \iint_{(t_m, t_{m+1}] \times M_0} \psi_k^s(\pi, \langle \pi, w_{t_{m+1}}^s(k, f_{m+1}) \rangle) H(\mathrm{d}s, \mathrm{d}\pi) \right\}.$$

By (3.11) and the induction hypothesis we have

$$\lim_{k\to\infty} Q_{r,\mu}^{(k)} \exp \sum_{i=1}^{m+1} \langle Y_{t_i}^{(k)}, -f_i\rangle$$

$$= Q_{r,\mu} \prod_{i=1}^{m-1} \exp\langle Y_{t_i}, -f_i\rangle \cdot \exp\langle Y_{t_m}, -f_m - w_{t_{m+1}}^{t_m}(f_{m+1})\rangle$$

$$\cdot \exp\left\{- \iint_{(t_m, t_{m+1}]\times M_0} \psi^s(\pi, \langle \pi, w_{t_{m+1}}^s(f_{m+1})\rangle) H(\mathrm{d}s, \mathrm{d}\pi)\right\}$$

$$= Q_{r,\mu} \exp \sum_{i=1}^{m+1} \langle Y_{t_i}, -f_i\rangle,$$

and the convergence is uniform in $f_1, \ldots, f_{m+1} \in B(E)_a^+$. Theorem 3.5 is proved. □

If E is a compact metric space, then the n-dimensional product topological space $M^n = \{(\mu_1, \ldots, \mu_n): \mu_1, \ldots, \mu_n \in M\}$ is locally compact and separable, and the function class

$$F(\mu_1, \ldots, \mu_n) = \exp \sum_{i=1}^{n} \langle \mu_i, -f_i\rangle, \quad f_i \in C(E)^{++},$$

is convergence determining. Thus (3.10) implies that $Y^{(k)}$ converges to Y in finite dimensional distributions.

3.4. In this subsection, we prove a result on the weak convergence in space $D(\mathbb{R}^+, M)$ of homogeneous MBDI-processes. Let $Y^{(k)} = (Y_t^{(k)}, t \geq 0)$ be a sequence of MBDI-processes with parameters $(W^{(k)}, \alpha_k\eta_k, h_k, k^{-1})$, where for each k,

- $W_t^{(k)}: f \mapsto w_t^{(k)}$ is a strongly continuous Ψ-semigroup on $C(E)^{++}$;
- α_k is a positive number;
- η_k is a finite measure on M_0;
- $h_k(\pi, \cdot)$, for every $\pi \in M_0$, is a probability generating function with $h_k(\pi, z)$ jointly continuous in (π, z).

The transition probabilities $Q_\mu^{(k)}$ of $Y^{(k)}$ are defined by

$$Q_\mu^{(k)} \exp\langle Y_t^{(k)}, -f\rangle$$

$$= \exp\left\{-\langle \mu, w_t^{(k)}\rangle - \int_0^t \mathrm{d}s \int_{M_0} \psi_k(\pi, \langle \pi, w_s(k)\rangle)\eta_k(\mathrm{d}\pi)\right\}, \tag{3.12}$$

with

$$w_t(k, x) = k[1 - \exp\{-k^{-1} w_t^{(k)}(x)\}] \tag{3.13}$$

and

$$\psi_k(\pi, \lambda) = \alpha_k[1 - h_k(\pi, 1 - \lambda/k)], \quad 0 \leq \lambda \leq k. \tag{3.14}$$

Clearly $Y^{(k)}$ has a strongly continuous Feller semigroup on $C_0(M)$, so we can assume it has sample paths in $D(\mathbb{R}^+, M)$.

Theorem 3.7. *Let $W_t : f \mapsto w_t(f)$ be a strongly continuous Ψ-semigroup on $C(E)^{++}$, and let $Y = (Y_t, t \geq 0)$ be an MBI-process in $D(\mathbb{R}^+, M)$ with parameters (W, η, ψ) with initial distribution Λ. Suppose that*

(3.F) *for every $f \in C(E)^{++}$, $w_t^{(k)}(x, f) \to w_t(x, f)$ $(k \to \infty)$ uniformly in (t, x) on each set $[0, l] \times E$;*

(3.G) *$\eta_k \to \eta$ weakly;*

(3.H) *$\psi_k(\pi, \lambda) \to \psi(\pi, \lambda)$ uniformly in (π, λ) on each set $M_0 \times [0, l]$;*

(3.I) *$Y_0^{(k)}$ has limiting distribution Λ.*

Then $Y^{(k)}$ converges weakly to Y in the space $D(\mathbb{R}^+, M)$ as $k \to \infty$.

Proof. By Theorem 2.5 of Ethier and Kurtz (1986, p. 167), it is sufficient to prove

$$\sup_{\mu \in M} |Q_\mu^{(k)} \exp\langle Y_t^{(k)}, -f\rangle - Q_\mu \exp\langle Y_t, -f\rangle| \to 0 \quad (k \to \infty) \tag{3.15}$$

for every fixed $f \in C(E)^{++}$ and $t \geq 0$. Let $2a = \inf_x w_t(x)$. By (3.F), there is a k_1 such that $w_t^{(k)} \geq a$ for all $k > k_1$.

Suppose $0 < \varepsilon < 1$. If $\mu(E) \geq a^{-1} \log \varepsilon^{-1}$, we have

$$\begin{aligned} &|Q_\mu^{(k)} \exp\langle Y_t^{(k)}, -f\rangle - Q_\mu \exp\langle Y_t, -f\rangle| \\ &\quad \leq \mathrm{e}^{-\langle \mu, w_t^{(k)}\rangle} + \mathrm{e}^{-\langle \mu, w_t\rangle} < 2\varepsilon \quad \text{for } k > k_1. \end{aligned}$$

If $\mu(E) < a^{-1} \log \varepsilon^{-1}$, then

$$\begin{aligned} &|Q_\mu^{(k)} \exp\langle Y_t^{(k)}, -f\rangle - Q_\mu \exp\langle Y_t, -f\rangle| \\ &\quad \leq |\langle \mu, w_t^{(k)}\rangle - \langle \mu, w_t\rangle| \\ &\qquad + \left| \int_0^t \mathrm{d}s \int_{M_0} \psi_k(\pi, \langle \pi, w_s(k)\rangle) \eta_k(\mathrm{d}\pi) \right. \\ &\qquad \left. - \int_0^t \mathrm{d}s \int_{M_0} \psi(\pi, \langle \pi, w_s\rangle) \eta(\mathrm{d}\pi) \right| \\ &\quad \leq a^{-1} \|w_t^{(k)} - w_t\| \log \varepsilon^{-1} + \varepsilon_1(k) + \varepsilon_2(k) + \varepsilon_3(k), \end{aligned}$$

where

$$\varepsilon_1(k) = \int_0^t \mathrm{d}s \int_{M_0} |\psi_k(\pi, \langle \pi, w_s(k)\rangle) - \psi(\pi, \langle \pi, w_s(k)\rangle)| \eta_k(\mathrm{d}\pi),$$

$$\varepsilon_2(k) = \int_0^t \mathrm{d}s \int_{M_0} |\psi(\pi, \langle \pi, w_s(k)\rangle) - \psi(\pi, \langle \pi, w_s\rangle)| \eta_k(\mathrm{d}\pi),$$

$$\varepsilon_3(k) = \int_0^t \left| \int_{M_0} \psi(\pi, \langle \pi, w_s\rangle) \eta_k(\mathrm{d}\pi) - \int_{M_0} \psi(\pi, \langle \pi, w_s\rangle) \eta(\mathrm{d}\pi) \right| \mathrm{d}s.$$

By (3.F), there exists k_2 such that

$$a^{-1} \|w_t^{(k)} - w_t\| \log \varepsilon^{-1} < \varepsilon \quad \text{for } k > k_2.$$

(3.F) also implies $w_t(k, x) \to w_t(x)$ boundedly and uniformly on each set $[0, l] \times E$. Then (3.G) and (3.H) yield the existence of k_3 such that $\varepsilon_1(k) + \varepsilon_2(k) < \varepsilon$ for $k > k_3$. By (3.G) and the dominated convergence theorem there is a k_4 such that $\varepsilon_3(k) < \varepsilon$ for $k > k_4$. Thus (3.15) follows. □

4. Particle systems and superprocesses

4.1. As usual, let E be a topological Lusin space. Suppose we have ξ, K, H and h given by (2.C), (2.D), (3.B) and (3.D) respectively. Assume that

(4.A) $g^s(x, z) = \sum_{i=0}^{\infty} p_i^s(x) z^i$, for every $(s, x) \in \mathbb{R} \times E$, is a probability generating function with the $p_i^s(x)$ and $\sum_{i=1}^{\infty} i p_i^s(x)$ belonging to $B(\mathbb{R} \times E)^+$.

A branching particle system with immigration with parameters (ξ, K, g, H, h) is described as follows:

(i) The particles in E move according to the law of ξ.

(ii) For a particle which is alive at time r and follows the path $(\xi_t, t \geq r)$, the conditional probability of survival during the time interval $[r, s)$ is $e^{-K(r,s)}$.

(iii) When a particle dies at time s at point $x \in E$, it gives birth to a random number of offspring at the death site according to the generating function $g^s(x, \cdot)$.

(iv) The entry times and distributions of new particles immigrating to E are governed by a Poisson random measure with intensity $H(ds, d\pi)$.

(v) The generating function $h^s(\pi, \cdot)$ gives the distribution of the number of new particles entering E at time s with distribution π.

For $t \in \mathbb{R}$, let $Y_t(B)$ be the number of particles of the system in set $B \in \mathscr{B}(E)$ at time t. Under standard independence hypotheses, $(Y_t, t \in \mathbb{R})$ form a Markov process in space M_1. (Note that the state space of the particle system is different from that of the MBDI-process.) The rigorous construction of the process can be reduced to constructing a branching particle system with parameters (ξ, K, g) generated by a single particle, which was given in Dynkin (1991). The transition probabilities $Q_{r,\sigma}$ of $(Y_t, t \in \mathbb{R})$ are determined by the Laplace functionals (cf. (3.6)):

$$Q_{r,\sigma} \exp\langle Y_t, -f\rangle = \exp\left\{-\langle \sigma, v_t^r\rangle - \iint_{(r,t] \times M_0} [1 - h^s(\pi, \langle \pi, e^{-v_t^s}\rangle)] H(ds, d\pi)\right\},$$

$$f \in B(E)^+, \quad \sigma \in M_1, \quad r \leq t, \tag{4.1}$$

where $v_t^r(x) \equiv v_t^r(x, f)$ is the unique positive solution of

$$e^{-v_t^r(x)} = \Pi_{r,x}\, e^{-f(\xi_t) - K(r,t)} + \Pi_{r,x} \int_r^t e^{-K(r,s)} g^s(\xi_s, e^{-v_t^s(\xi_s)}) K(ds). \tag{4.2}$$

This equation arises as follows: If we start one particle at time r at point x, this particle moves following a path of ξ and does not branch before time t with probability $e^{-K(r,t)}$ (first term on the right-hand side), or it splits at time $s \in (r, t]$ with probability $e^{-K(r,s)}K(ds)$ according to $g^s(\xi_s, \cdot)$ and all the offspring evolve independently after birth in the same fashion (second term). By Lemma 2.3 of Dynkin (1991), (4.2) is equivalent to

$$\Pi_{r,x}\, e^{-f(\xi_t)} - e^{-v_t^r(x)} = \Pi_{r,x} \int_r^t [e^{-v_t^s(\xi_s)} - g^s(\xi_s, e^{-v_t^s(\xi_s)})] K(ds). \tag{4.3}$$

4.2. Let $Y(k) = \{Y_t(k), t \in \mathbb{R}\}$ be a sequence of branching particle systems with immigration with parameters $(\xi, \gamma_k K, g_k, \alpha_k H, h_k)$, where $\alpha_k \geq 0$, $\gamma_k \geq 0$, $k = 1, 2, \ldots$. Then

$$Y^{(k)} = \{Y_t^{(k)} := k^{-1} Y_t(k), t \in \mathbb{R}\}$$

is a Markov process in space M_k with transition probabilities $Q_{r,\sigma_k}^{(k)}$ determined by

$$Q_{r,\sigma_k}^{(k)} \exp\langle Y_t^{(k)}, -f\rangle$$
$$= \exp\left\{-\langle \sigma_k, k v_t^r(k)\rangle - \iint_{(r,t]\times M_0} \psi_k^s(\pi, \langle \pi, w_t^s(k)\rangle) H(ds, d\pi)\right\},$$
$$f \in B(E)^+, \quad \sigma_k \in M_k, \quad r \leq t, \tag{4.4}$$

where $\psi_k^s(\pi, \lambda)$ is given by (3.9), $v_t^r(k, x) \equiv v_t^r(k, x, f)$ satisfies

$$\Pi_{r,x}\, e^{-f(\xi_t)/k} - e^{-v_t^r(k,x)}$$
$$= \Pi_{r,x} \int_r^t \gamma_k [e^{-v_t^s(k,\xi_s)} - g_k^s(\xi_s, e^{-v_t^s(k,\xi_s)})] K(ds) \tag{4.5}$$

and

$$w_t^s(k, x) \equiv w_t^s(k, x, f) = k[1 - e^{-v_t^s(k,x,f)}]. \tag{4.6}$$

Let $Q_{r,\mu_k}^{(k)}$ denote the conditional law of $(Y_t^{(k)}, t \geq r)$ given $Y_r^{(k)} = k^{-1}\sigma(k\mu)$, where μ belongs to M and $\sigma(k\mu)$ is a Poisson random measure with intensity $k\mu$. Then

$$Q_{r,\mu_k}^{(k)} \exp\langle Y_t^{(k)}, -f\rangle$$
$$= \exp\left\{-\langle \mu, w_t^r(k)\rangle - \iint_{(r,t]\times M_0} \psi_k^s(\pi, \langle \pi, w_t^s(k)\rangle) H(ds, d\pi)\right\}. \tag{4.7}$$

It is easy to check that $w_t^r(k)$ satisfies

$$w_t^r(k, x) + \Pi_{r,x} \int_r^t \phi_k^s(\xi_s, w_t^s(k, \xi_s)) K(ds) = \Pi_{r,x} k[1 - e^{-f(\xi_t)/k}], \tag{4.8}$$

with

$$\phi_k^s(x, \lambda) = k\gamma_k [g_k^s(x, 1 - \lambda/k) - (1 - \lambda/k)], \quad 0 \leq \lambda \leq k. \tag{4.9}$$

For the sequence (4.9) we note

$$\bar{b}_k = \sup_{s,x} \left| \frac{d}{d\lambda} \phi_k^s(x, \lambda) \right|_{\lambda=0}. \tag{4.10}$$

Lemma 4.1. (i) *Suppose that*

(4.B) $\phi_k^s(x,\lambda)\to\phi^s(x,\lambda)$ $(k\to\infty)$ *uniformly on each set* $\mathbb{R}\times E\times[0,l]$;

(4.C) $\phi^s(x,\lambda)$ *is Lipschitz in* λ *uniformly on each set* $\mathbb{R}\times E\times[0,l]$.

Then $\phi^s(x,\lambda)$ *has the representation* (2.E).

(ii) *If* $\phi^s(x,\lambda)$ *is given by* (2.E), *then it satisfies* (4.C) *and there is a sequence* $\phi_k^s(x,\lambda)$ *in form* (4.9) *such that* (4.B) *holds and*

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\phi_k^s(x,\lambda)|_{\lambda=0}=b^s(x),\quad s\in\mathbb{R},\ x\in E. \tag{4.11}$$

Proof. Assertion (i) follows easily by a result of Li (1991), so we shall prove (ii) only. Suppose that $\phi^s(x,\lambda)$ is given by (2.E). (4.C) holds clearly. Let

$$\gamma_{1,k}=1+\sup_{s,x}\int_0^\infty u(1-\mathrm{e}^{-ku})m^s(x,\mathrm{d}u)$$

and

$$g_{1,k}^s(x,z)=z+k^{-1}\gamma_{1,k}^{-1}\int_0^\infty[\mathrm{e}^{ku(z-1)}-1+ku(1-z)]m^s(x,\mathrm{d}u).$$

It is easy to check that

$$\begin{aligned}\phi_{1,k}^s(x,\lambda)&:=k\gamma_{1,k}[g_{1,k}^s(x,1-\lambda/k)-(1-\lambda/k)]\\&=\int_0^\infty(\mathrm{e}^{-\lambda u}-1+\lambda u)m^s(x,\mathrm{d}u).\end{aligned}$$

Let

$$\bar b=\sup_{s,x}|b^s(x)|,\qquad \bar c=\sup_{s,x}c^s(x).$$

Assuming $\gamma_{2,k}:=\bar b+2k\bar c>0$ and setting

$$g_{2,k}^s(x,z)=\begin{cases}z+\gamma_{2,k}^{-1}[b^s(x)(1-z)+kc^s(x)(1-z)^2], & \text{if } b^s(x)\geq 0,\\ \gamma_{2,k}^{-1}[\frac12\bar b(1+z^2)+\frac12 b^s(x)(1-z^2)+kc^s(x)(1-z)^2+2k\bar c z], & \text{if } b^s(x)<0,\end{cases}$$

we have

$$\begin{aligned}\phi_{2,k}^s(x,\lambda)&:=k\gamma_{2,k}[g_{2,k}^s(x,1-\lambda/k)-(1-\lambda/k)]\\&=\begin{cases}b^s(x)\lambda+c^s(x)\lambda^2, & \text{if } b^s(x)\geq 0,\\ b^s(x)\lambda+c^s(x)\lambda^2+(2k)^{-1}[\bar b-b^s(x)]\lambda^2, & \text{if } b^s(x)<0.\end{cases}\end{aligned}$$

Finally we let

$$\gamma_k=\gamma_{1,k}+\gamma_{2,k}\quad\text{and}\quad g_k=\gamma_k^{-1}(\gamma_{1,k}g_{1,k}+\gamma_{2,k}g_{2,k}).$$

Then the sequence $\phi_k^s(x,\lambda)$ defined by (4.9) is equal to $\phi_{1,k}^s(x,\lambda)+\phi_{2,k}^s(x,\lambda)$ that satisfies (4.B) and (4.11). $\square$

Lemma 4.2. *If conditions* (4.B) *and* (4.C) *are fulfilled and if*

(4.D) $\sup_k \bar{b}_k < \infty$,

then $w_t^r(k, x, f)$, *and hence* $kv_t^r(k, x, f)$, *converge boundedly and uniformly on each set* $[u, t] \times E \times B(E)_a^+$ *of* (r, x, f) *to the unique bounded positive solution of equation* (2.6).

Proof. Since $-(\mathrm{d}/\mathrm{d}\lambda)\phi_k^s(x, \lambda) \leq \bar{b}_k$, (4.8) implies that

$$w_t^r(k, x) \leq \|f\| + \bar{b}_k \Pi_{r,x} \int_r^t w_t^s(k, \xi_s) K(\mathrm{d}s). \tag{4.12}$$

By the generalized Gronwall's inequality proved by Dynkin (1991), we get

$$w_t^r(k, x) \leq \|f\| \Pi_{r,x} \, \mathrm{e}^{\bar{b}_k K(r,t)}. \tag{4.13}$$

Using (4.8) and (4.13), the convergence of $w_t^r(k)$ is proved in the same way as Lemma 3.3 of Dynkin (1991). The convergence of $kv_t^r(k)$ follows by (4.6). □

Based on Lemmas 4.1 and 4.2, the following result can be obtained similarly as Theorem 3.5.

Theorem 4.3. (i) *Let* $Y^{(k)}$ *be the sequence of renormalized branching particle systems with immigration defined by* (4.7), *and let* Y *be the* (ξ, K, ϕ, H, ψ)*-superprocess. Assume that conditions* (3.E) *and* (4.B), (4.C), (4.D) *are satisfied. Then for every* $\mu \in M$, $r \leq t_1 < \cdots < t_n$ *and* $a \geq 0$,

$$Q_{r,\mu_k}^{(k)} \exp \sum_{i=1}^n \langle Y_{t_i}^{(k)}, -f_i \rangle \to Q_{r,\mu} \exp \sum_{i=1}^n \langle Y_{t_i}, -f_i \rangle \quad (k \to \infty) \tag{4.14}$$

uniformly in $f_1, \ldots, f_n \in B(E)_a^+$.

(ii) *To each* (ξ, K, ϕ, H, ψ)*-superprocess* Y *there corresponds a sequence of branching particle systems with immigration* $Y^{(k)}$ *satisfying* (4.14). □

Suppose that each particle in the kth system is weighted k^{-1}. (4.14) states that the mass distribution of the particle system approximates to the process Y when the single mass becomes small and the particle population becomes large. Typically, $\gamma_k \to \infty$ and $\alpha_k \to \infty$, which mean that the rates of the branching and the immigration get high. It is also possible to prove a result on the weak convergence in space $D(\mathbb{R}^+, M)$ of the branching system of particles with immigration. The discussions are similar to those of Section 3 and left to the reader.

5. Transformations of the measure space

By transformations of the state space M, large classes of MBI-processes that may take infinite (but σ-finite) values can be obtained from the MBI-processes with finite values that we have discussed in Sections 3 and 4.

For $\rho \in B(E)^{++}$ we let

$M^{\rho} = \{\mu\colon \mu$ is a measure on $(E, \mathscr{B}(E))$ such that $\langle \mu, \rho\rangle < \infty\}$,

$M_0^{\rho} = \{\tau\colon \tau \in M^{\rho}$ and $\langle \tau, \rho\rangle = 1\}$.

Suppose that $W_t^r\colon f \mapsto w_t^r(f)$ is a cumulant semigroup on $B(E)^+$ such that

(5.A) for every $f \in B(E)^+$ and $u \leq t \in \mathbb{R}$, the function $\rho^{-1}(x) w_t^r(x, \rho f)$ of (r, x) restricted to $[u, t] \times E$ belongs to $B([u, t] \times E)^+$.

We define the operators $\hat{W}_t^r\colon f \mapsto \hat{w}_t^r(f)$ on $B(E)^+$ by

$$\hat{w}_t^r(f) = \rho^{-1} w_t^r(\rho f) \tag{5.1}$$

(cf. El-Karoui and Roelly-Coppoletta, 1989). It is easy to see that $\hat{W}_t^r$, $r \leq t \in \mathbb{R}$, also form a cumulant semigroup. If $(\hat{Y}_t, t \in \mathbb{R})$ is an MBI-process in M with parameters $(\hat{W}, H, \psi)$ (Definition 3.1), then

$$Y = (Y_t := \rho^{-1} \hat{Y}_t, t \in \mathbb{R}) \tag{5.2}$$

is an MBI-process in the space M^{ρ} with transition probabilities $Q_{r,\mu}$ determined by

$$Q_{r,\mu} \exp\langle Y_t, -f\rangle = \exp\left\{-\langle \mu, w_t^r\rangle - \iint_{(r,t]\times M_0^{\rho}} \psi_{\rho}^s(\tau, \langle \tau, w_t^s\rangle) H_{\rho}(\mathrm{d}s, \mathrm{d}\tau)\right\},$$

$$f \in B(E)^+, \quad \mu \in M^{\rho}, \quad r \leq t, \tag{5.3}$$

where

$$H_{\rho}(\mathrm{d}s, \mathrm{d}\tau) = H(\mathrm{d}s, \mathrm{d}\rho\tau) \quad \text{and} \quad \psi_{\rho}^s(\tau, \lambda) = \psi^s(\rho\tau, \lambda).$$

Example 5.1. Suppose that $0 < \beta \leq 1$ and that Π_t is the semigroup of the d-dimensional Brownian motion. Then equation

$$w_t + \int_0^t \Pi_{t-s}(w_s)^{1+\beta}\, \mathrm{d}s = \Pi_t f, \quad t \geq 0, \tag{5.4}$$

defines a homogeneous cumulant semigroup $W_t\colon f \mapsto w_t$. For $p > d$, let $\rho(x) = (1 + |x|^p)^{-1}$, $x \in \mathbb{R}^d$, and let $M_p(\mathbb{R}^d) = \{\mu\colon \mu$ is a Borel measure on $\mathbb{R}^d$ such that $\langle \mu, \rho\rangle < \infty\}$. Iscoe (1986) showed that W_t satisfies condition (5.A). Assume $0 < \theta \leq 1$ and $\lambda \in M_p(\mathbb{R}^d)$. Then formula

$$Q_{\mu} \exp\langle Y_t, -f\rangle = \exp\left\{-\langle \mu, w_t\rangle - \int_0^t \langle \lambda, w_s\rangle^{\theta}\, \mathrm{d}s\right\} \tag{5.5}$$

defines an MBI-process $Y = (Y_t, Q_{\mu})$ in the space $M_p(\mathbb{R}^d)$. When $\beta = \theta = 1$, Y has continuous sample paths almost surely (in a suitable topology in $M_p(\mathbb{R}^d)$; see Konno and Shiga, 1988).

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